Ramsey Stochastic Model via Multistage Stochastic Programming

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Abstract. Ramsey model belongs to "classical" economic dynamic models. It has been (1928) originally constructed (with a farmer's interpretation) in a deterministic discrete setting. To solve it Lagrangean or dynamic programming techniques can be employed. Later, this model has been generalized to a stochastic version. Time horizon in the original deterministic model as well as in modified stochastic one can be considered finite or infinite.

We plan to deal with the stochastic model and finite horizon. However, in spite of the classical approach to analyze it we employ a stochastic programming technique. This approach gives a possibility to employ well known results on stability and empirical estimates also in the case of the Ramsey model. However, first we introduce some confidence intervals. To obtain the new assertions we restrict our consideration mostly to the case when the "underlying" random element follows autoregressive (or at least Markov) sequence.

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1 Introduction

1.1 Deterministic Approach

To recall a "classical" deterministic Ramsey farmer model (in a discrete setting with a finite horizon) let the time interval be divided into equal length subintervals denoted by $t = 0, 1, \dots T$; K_t and N_t denote the invested amounts of capital and labor in period t; Y_t denote the amount of corn production in the period t; C_t denote the consumption. Furthermore, let F, U denote production and utility functions both defined on the corresponding subsets of R^2 and R^{T+1} of the Euclidean spaces. If we assume that

$$\begin{array}{lcl} Y_t & = & F(N_t, K_t), \\ f(K_t) & := & F(N, K_t) + (1-\delta)K_t, \quad N_t = N, \quad N \in \mathcal{N}\,\text{given}, \quad t = 0, \dots, T, \end{array} \tag{1}$$

where $\delta \in \langle 0, 1\rangle$ is the rate of capital depreciation, $\mathcal{N}$ denotes the set of natural numbers, then we can recall a classical Ramsey problem ([2]) as the following deterministic optimization problem:

Find

$$\begin{array}{l} \varphi := \varphi(K_0, F, U) = \max\limits_{\{C_0, \dots, C_T\}} U(C_0, \dots, C_T), \\ (C_0, C_1, \dots, C_T) := ((K_1, C_0), (K_2, C_1), \dots, (K_{T+1}, C_T)) \in \mathcal{K}(f, F, K_0), \end{array} \tag{2}$$

where

$$\begin{array}{llcl} \mathcal{K}(f, F, K_0) = \{(K_{t+1}, C_t): & K_{t+1} + C_t & \le & f(K_t), \\ & 0 & \le & C_t, \\ & 0 & \le & K_{t+1}, \qquad t = 0, \dots, T\}; \quad K_0\,\text{knowm}. \end{array} \tag{3}$$

If K_0 and F, f are known, then it is possible to determine (under general conditions) the optimal (K_{t+1}, C_t) for every $t = 0, \dots, T$.

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Remark. In the classical Ramsey model it is assumed: There is no free lunch; $0 = F(0, 0)$, $f(0) = 0$; F is a strictly increasing in both of its arguments, concave (i.e. we rule out increasing returns to scale) and twice continuously differentiable.

Furthermore, evidently it is reasonable to assume that the production function F is nonnegative. According to this fact, denoting by $\mathcal{K}$ a set that cover possible values K_t, $t = 0, \ldots, T$ and assuming that there exist functions $\underline{F}$, $\bar{F}$ (or constants) defined on R^2 such that

$$\underline{F}(N, K) \le F(N, K) \le \bar{F}(N, K), \quad t = 0, \ldots, T, \quad K \in \mathcal{K},$$

we can see that

$$\begin{array}{ccccc} \mathcal{K}(f, \underline{F}, K_0) & \subset & \mathcal{K}(f, F, K_0) & \subset & \mathcal{K}(f, \bar{F}, K_0), \\ \varphi(K_0, \underline{F}, U) & \le & \varphi(K_0, F, U) & \le & \varphi(K_0, \bar{F}, U). \end{array} \tag{4}$$

1.2 Stochastic Approach

The problem (2), (3) is a deterministic dynamic optimization problem (with a finite time horizon) not depending on any random factor. However, (in applications) very often the production function depends on random elements $\xi^t := \xi^t(\omega)$, $t = 0, \ldots, T$ defined (from the mathematical point of view) on a probability space $(\Omega, \mathcal{S}, P)$. We try to analyze this stochastic case successively, first, we assume:

i.1 There exist nonnegative functions $\bar{F}_t$, $\bar{f}_t$, $t = 0, \ldots, T$, defined on R^3 and R^2 such that

$$\begin{array}{lll} F(N, K_t) & := & \bar{F}_t(N, K_t, \xi^t), \quad N \in \mathcal{N} \text{ given}, \\ \bar{f}(K_t) & := & \bar{f}_t(K_t, \xi^t) \quad = \bar{F}_t(N, K_t, \xi^t) + (1-\delta)K_t, \qquad t = 0, \ldots, T. \end{array} \tag{5}$$

The problem (2), (3) is in this case replaced by the problem:

Find

$$\varphi := \bar{\varphi}(K_0, \bar{F}^T, U) = \max_{\{C_0, \ldots, C_T\}} U(C_0, \ldots, C_T) \tag{6}$$

subject to the constraints set

$$\begin{array}{rlll} \bar{\mathcal{K}}(\bar{f}, \bar{F}^T, K_0) := \{(K_{t+1}, C_t): & K_{t+1} + C_t & \le & \bar{f}_t(K_t, \xi^t), \\ & 0 & \le & K_{t+1}, \\ & 0 & \le & C_t, \qquad t = 0, 1, \ldots, T\}, \end{array} \tag{7}$$

$$K_0 \text{ known}, \quad \bar{F}^t := \bar{F}^t = (\bar{F}_0, \bar{F}_1, \ldots, , \bar{F}_t), \ t = 0, \ldots, T.$$

Furthermore, if we can assume that for an $\alpha \in \langle 0, 1)$ there exist nonnegative functions $\bar{F}_{t,\alpha}$, $\bar{F}_t^{\alpha}$, $t = 0, \ldots, T$ defined on R^2 (not depending on a random element) such that

$$P\{\omega : \bar{F}_{t,\alpha}(N, K_t) \le \bar{F}_t(N, K_t, \xi^t) \le \bar{F}_t^{\alpha}(N, K_t), \quad t = 0, \ldots, T\} \ge 1 - \alpha, \tag{8}$$

then employing (1), (5) and setting $f = f(K_t) := f_t(F_t)$, $\quad \bar{f}_t(K_t, \xi^t) := \bar{f}_t(\bar{F}_t, \xi^t)$, $\quad t = 0, 1, \ldots, T$, we obtain

$$P\{\omega : f_t(\bar{F}_{t,\alpha}) \le \bar{f}_t(\bar{F}_t, \xi^t) \le f_t(\bar{F}_t^{\alpha}), \quad t = 0, \ldots, T\} \ge 1 - \alpha. \tag{9}$$

According to (7), the definition of $\mathcal{K}$ and φ (in which we replaced successively F by $F_{t,\alpha}$ and F_t^{α}, $t = 0, \ldots, T$) we obtain

$$P\{\omega : \mathcal{K}(f, \bar{F}_{\alpha}^T, K_0) \subset \bar{\mathcal{K}}(\bar{f}, \bar{F}^T, K_0) \subset \mathcal{K}(f, \bar{F}^{T,\alpha}, K_0) \quad \} \ge 1 - \alpha. \tag{10}$$

where $\bar{F}_{\alpha}^T = (\bar{F}_{0,\alpha}, \ldots, \bar{F}_{T,\alpha})$, $\quad \bar{F}^{T,\alpha} = (\bar{F}_0^{\alpha}, \ldots, \bar{F}_T^{\alpha})$.

Proposition 1. Let i.1 be fulfilled. Let, moreover, for an $\alpha \in \langle 0, 1\rangle$ the relation (8) be fulfilled, then

$$P\{\omega : \bar{\varphi}(K_0, \bar{F}_{\alpha}^T, U) \le \bar{\varphi}(K_0, \bar{F}^T, U) \le \bar{\varphi}(K_0, \bar{F}^{T,\alpha}, U)\} \ge 1 - \alpha,$$

Proposition 1 gives a confidence interval for the optimal value of the problem (6) and (7) under the assumption that the realization of $(\xi^0, \ldots, \xi^T)$ will be known at the beginning of the problem. Consequently, it corresponds to some type of anticipative solution. Of course, this assumption is not realistic and the following section better corresponds to a real situation. To this end, first we shall construct the corresponding multistage programming problem and restrict our consideration to Markov type dependence and furthermore to an autoregressive type of "underlying" sequences. Employing this special cases we construct confidence intervals and furthermore we sketch a possibility to employ stochastic programming results achieved for stability and empirical estimates to the dynamic Ramsey model with finite horizon.

2 Multistage Stochastic Programming Approach

We assume:

i.2 The sequences of decisions (K_{t+1}, C_t) and random elements realizations ξ^t follows the relation

$$\begin{array}{l} K_0 \longrightarrow \xi^0 \longrightarrow (K_1, C_0) \longrightarrow \xi^1 \longrightarrow (K_2, C_1) \longrightarrow \quad \ldots\ldots \longrightarrow \\ \longrightarrow \xi^{T-1} \longrightarrow (K_T, C_{T-1}), \longrightarrow \xi^T \longrightarrow (K_{T+1}, C_T), \qquad K_0 \quad \text{known}, \end{array}$$

i.3 the utility function U is additive with a discount factor $\beta \in \langle 0, 1 \rangle$,

$$U(C_0, \ldots, C_T) := U^{\beta}(K_0, C_0, C_1, \ldots, C_T) = \sum_{t=0}^{T} \beta^t u_t(C_t), \tag{11}$$

where u_t is an increasing utility function corresponding to the individual time point $t \in \{0, 1, \ldots, T\}$,

i.4 it is reasonable to determine a solution w.r.t. the mathematical expectation of the utility function.

According to i.2, the decision (K_{t+1}, C_t), $t = 0, \ldots, T$ is evidently nonanticipative; it means that for every $t \in \{0, \ldots, T\}$, (K_{t+1}, C_t) can depend on $\xi^0, \ldots, \xi^t, K_0, \ldots, K_t, C_0, \ldots, C_{t-1}$, but it can not depend on $\xi^{t+1}, \ldots, \xi^T, \quad K_{t+2}, \ldots, K_{T+1}, \quad C_{t+1}, \ldots, C_T$. If we define the mappings $\bar{\mathcal{K}}_t$ by

$$\begin{array}{rlll} \bar{\mathcal{K}}_t(K_t, \xi^t) := \mathcal{K}_t(\bar{f}, \bar{F}_t, K_0) = \{(K_{t+1}, C_t): & K_{t+1} + C_t & \leq & \bar{f}_t(K_t, \xi^t), \\ & 0 & \leq & K_{t+1}, \\ & 0 & \leq & C_t \qquad \}, \qquad t = 0, 1, \ldots, T, \end{array} \tag{12}$$

and if symbols F^{ξ^0}, $F^{\xi^t|\bar{\xi}^{t-1}}$ denote the distribution function of ξ^0 and the conditional distribution functions of ξ^t conditioned by $\bar{\xi}^{t-1}$, then we can set to the stochastic Ramsey model (with the finite horizon T) the following multistage ($T+1-$ stage) stochastic programming problem:

Find

$$\bar{\varphi}_{\mathcal{F}}(K_0, \bar{F}^T, U) = \mathsf{E}_{F^{\xi^0}} g^0_{\mathcal{F}}(K_0, \bar{F}^T, U, \xi^0),$$

$$g^0_{\mathcal{F}}(K_0, \bar{F}^T, U, \xi^0) = \max_{(K_1, C_0) \in \bar{\mathcal{K}}_0(K_0, \xi^0)} \mathsf{E}_{F^{\xi^1|\xi^0}} g^1_{\mathcal{F}}(\bar{K}^1, \bar{C}^0, \bar{F}^0, \bar{\xi}^1),$$

where the function $g^1_{\mathcal{F}}(\bar{K}^1, \bar{C}^0, \bar{F}^0, \bar{\xi}^1)$ is defined recursively

$$g^t_{\mathcal{F}}(\bar{K}^t, \bar{C}^{t-1}, \bar{F}^{t-1}, \bar{\xi}^t) = \max_{(K_{t+1}, C_t) \in \bar{\mathcal{K}}_t(K_t, \xi^t)} [\sum_{v=0}^{t} \beta^v u_v(C_v) + \mathsf{E}_{F^{\xi^{t+1}|\bar{\xi}^t}} g^{t+1}_{\mathcal{F}}(\bar{K}^{t+1}, \bar{C}^t, \bar{F}^t, \bar{\xi}^{t+1})],$$

$t = 1, \ldots, T-1$ and,

$$g^{T-1}_{\mathcal{F}}(\bar{K}^{T-1}, \bar{C}^{T-2}, \bar{F}^{T-1}, \bar{\xi}^{T-1}) =$$

$$\max_{(K_T, C_{T-1}) \in \bar{\mathcal{K}}_{T-1}(K_{T-1}, \xi^{T-1})} [\sum_{v=0}^{T-1} \beta^{T-1} u_v(C_{T-1}) + \mathsf{E}_{F^{\xi^T|\bar{\xi}^{T-1}}} \max_{(K_{T+1}, C_T) \in \bar{\mathcal{K}}_T(K_T, \xi^T)} \beta^T u_T(C_T),$$

$$g^T_{\mathcal{F}}(\bar{K}^T, \bar{C}^{T-1}, \bar{F}^T, \bar{\xi}^T) := \max_{(K_{T+1}, C_T) \in \bar{\mathcal{K}}_T(K_T, \xi^T)} \sum_{t=1}^{T} \beta^t u(C_t), \qquad K_0 \quad \text{known}, \tag{13}$$

where the symbol $\mathcal{F}$ denotes a system of the probability (mostly conditional) measures:

$$\begin{array}{l} \mathcal{F} = \{F_{\xi^0}, \quad F^{\xi^{t+1}|\bar{\xi}^t}, t = 0, \ldots, T-1\} \\ \bar{\xi}^t = (\xi^0, \ldots, \xi^t), \quad \bar{K}^t = (K_0, \ldots, K_t), \quad \bar{C}^t = (C_0, \ldots, C_t). \end{array} \tag{14}$$

Remark. We assume that all symbols in (12) and (13) are meaningful.

Evidently, for every $t = 0, 1, \ldots, T$ optimal decision (K_{t+1}, C_t) is a function of ξ^t. Consequently, every realization of $g^0_{\mathcal{F}}(K_0, \bar{F}^T, U, \xi^0)$ is a random value with the realization denoting $\bar{\varphi}^R_{\mathcal{F}}(K_0, \bar{F}^T, U)$. Consequently, it is reasonable to construct confidence intervals, investigate stability and empirical estimates properties. To this end we assume:

i.5. • The random sequence $\{\xi^t\}_{t=-\infty}^{T+1}$ fulfils the Markov property,

- there exist constants (degenerate random values) $\bar{\xi}_{t,\,\alpha}$, $\bar{\xi}_t^{\alpha}$, $t = 0, \ldots T$, $\alpha \in \langle 0,\, 1)$ such that

$$
\begin{array}{lll}
P_{\xi^0|\xi^{-1}}\{\omega : \bar{\xi}_{0,\,\alpha} \le \xi^0(\omega) \le \bar{\xi}_0^{\alpha}\} & \ge & 1-\alpha, \\
P_{\xi^t|\xi^{t-1}}\{\omega : \bar{\xi}_{t,\,\alpha} \le \xi^t(\omega) \le \bar{\xi}_t^{\alpha}\} & \ge & 1-\alpha \quad \text{indenpedently on} \quad \xi^{t-1},\, t = 0, \ldots, T,
\end{array}
$$

- for every $\bar{F}_t$ the function (defined by the relation (5)) $\bar{f}_t(\bar{F}_t,\, \xi^t)$ is an increasing function of ξ^t.

and if we define systems $\mathcal{F}_{\alpha}$, $\mathcal{F}^{\alpha}$ (of degenerate distribution functions) by

$$
\begin{array}{lll}
\mathcal{F}_{\alpha} & = & \{F_{\xi^0}(z) = \delta_{\bar{\xi}_{0,\,\alpha}}(z),\ F^{\xi^{t+1}|\xi^t}(z) = \delta_{\bar{\xi}_{t,\,\alpha}}(z), \quad t = 0, 1, \ldots, T-1\}, \\
\mathcal{F}^{\alpha} & = & \{F_{\xi^0}(z) = \delta_{\bar{\xi}_0^{\alpha}}(z),\ F^{\xi^{t+1}|\bar{\xi}^t}(z) = \delta_{\bar{\xi}_t^{\alpha}}(z),\, t = 0, 1, \ldots, T-1\},
\end{array} \tag{15}
$$

then evidently we can obtain successively

$$
\begin{array}{l}
P_{\bar{\xi}^T}\{\omega : \bar{\mathcal{K}}_t(K_t,\, \bar{\xi}_{t,\alpha}) \subset \mathcal{K}_t(K_t,\, \xi^t) \subset \bar{\mathcal{K}}_t(K_t,\, \bar{\xi}_t^{\alpha}),\, t = 0, 1, \ldots, T\} \ge (1-\alpha)^{T+1}, \\
P_{\bar{\xi}^T}\{\omega : \bar{\varphi}^R_{\mathcal{F}_{\alpha}}(K_0,\, \bar{F}^T,\, U) \le \bar{\varphi}^R_{\mathcal{F}}(K_0,\, \bar{F}^T,\, U) \le \bar{\varphi}^R_{\mathcal{F}^{\alpha}}(K_0,\, \bar{F}^T,\, U)\} \ge (1-\alpha)^T.
\end{array} \tag{16}
$$

(A case of Markov dependence has been already analyzed for more general multistage problem in [3], [**?**]

In the case when it is possible to assume:

i.6 There exists a random sequence $\varepsilon^t := \varepsilon^t(\omega),\, t = \ldots,\ ,-1, 0, 1, \ldots$ defined on $(\Omega,\, \mathcal{S},\, P)$ and a positive, increasing, Lipschitz function $H(z)$ defined on R^1 such that

- ξ^{-1}, ε^t (defined on $(\Omega,\, \mathcal{S},\, P)$, $t = 0, 1, \ldots$ are stochastically independent random values,
- ε^t, $t = 0, 1\ldots$ are identically distributed. (We denote the distribution function corresponding to ε^1 by the symbol F^{ε} and suppose the realization ξ^{-1} to be known),

i.7 for $\alpha \in \langle 0,\, 1)$ there exist ε_{α}, ε^{α} such that $\quad P\{\omega :\, \varepsilon_{\alpha} \le \varepsilon^1(\omega) \le \varepsilon^{\alpha}\} \ge 1-\alpha$,

i.8 ξ^t follows random sequence such that $\xi^t = \varepsilon^t H(\xi^{t-1}),\ t = \ldots -1, 0, 1, \ldots,$

i.9 ξ^t follows random sequence such that $\xi^t = \varepsilon^t + H(\xi^{t-1}),\ t = \ldots -1, 0, 1, \ldots$

we can obtain "stronger" results.

Considering the assumptions i.6, i.7 and i.8 we obtain successively

$$
P_{\xi^t|\xi^{t-1}}\{\omega :\, \varepsilon_{\alpha} H(\xi^{t-1}) \le \varepsilon^t H(\xi^{t-1}) \le \varepsilon^{\alpha} H(\xi^{t-1})\} \ge 1-\alpha, \quad t = 0, \ldots, T, \tag{17}
$$

$$
\begin{array}{ll}
P_{\bar{\xi}^T|\xi^{-1}}\{\omega : & \varepsilon_{\alpha} H(\xi^{-1}) \le \varepsilon^0 H(\xi^{-1}) \le \varepsilon^{\alpha} H(\xi^{-1}) \ldots\ldots, \\
& \varepsilon_{\alpha} H(\xi^{T-1}) \le \varepsilon^T H(\xi^{T-1}) \le \varepsilon^{\alpha} H(\xi^{T-1})\} \ge (1-\alpha)^T,
\end{array}
$$

$$
\begin{array}{llll}
\bar{\mathcal{K}}_t(K_t,\, \xi^t) := \bar{\mathcal{K}}_t^{\varepsilon}(K_t,\, \xi^{t-1},\, \varepsilon^t) = \{(K_{t+1},\, C_t) : & K_{t+1} + C_t & \le \bar{f}_t(\bar{F}^t,\, \varepsilon^t H(\xi^{t-1})), & \\
& 0 & \le K_{t+1} & \\
& 0 & \le C_t & \}, \quad t = 0, \ldots, T.
\end{array}
$$

Defining in this special case systems $\mathcal{F}_{\alpha}(\varepsilon)$, $\mathcal{F}^{\alpha}(\varepsilon)$ of degenerate distribution functions by

$$
\begin{array}{lll}
\mathcal{F}_{\alpha}(\varepsilon) & = & \{F_{\xi^0|\xi^{-1}}(z) = \delta_{\bar{\xi}_{0,\,\alpha}}(z),\ F^{\xi^{t+1}|\bar{\xi}^t}(z) = \delta_{\xi_{t,\,\alpha}}(z), \quad t = 0, 1, \ldots, T-1\}, \quad \xi_{t,\,\alpha} = \varepsilon_{\alpha} H(\xi^{t-1}), \\
\mathcal{F}^{\alpha}(\varepsilon) & = & \{F_{\xi^0|\xi^{-1}}(z) = \delta_{\bar{\xi}_0^{\alpha}}(z),\ F^{\xi^{t+1}|\bar{\xi}^t}(z) = \delta_{\xi_t^{\alpha},}(z),\, t = 0, 1, \ldots, T-1\}, \quad \xi_t^{\alpha} = \varepsilon^{\alpha} H(\xi^{t-1})
\end{array} \tag{18}
$$

we can successively obtain

$$
\begin{array}{l}
P_{\xi^t|\xi^{t-1}}\{\omega :\, \bar{\mathcal{K}}_t(K_t,\, \varepsilon_{\alpha} H(\xi^{t-1}) \subset \mathcal{K}_t(K_t,\, \varepsilon^t H(\xi^{t-1}) \subset \bar{\mathcal{K}}_t(K_t,\, \varepsilon^{\alpha} H(\xi^{t-1}),\, \} \ge 1-\alpha, \quad t = 1, \ldots, T, \\
P_{\bar{\xi}^T}\{\omega : \bar{\varphi}^R_{\mathcal{F}_{\alpha}(\varepsilon)}(K_0,\, \bar{F}^T,\, U) \le \bar{\varphi}^R_{\mathcal{F}_{(\varepsilon)}}(K_0,\, \bar{F}^T,\, U) \le \bar{\varphi}^R_{\mathcal{F}^{\alpha}(\varepsilon)}(K_0,\, \bar{F}^T,\, U)\} \ge (1-\alpha)^T,
\end{array} \tag{19}
$$

where $\mathcal{F}(\varepsilon)$ is the corresponding system $\mathcal{F}$ defined by the assumption i.6., i.8.

Remark. In [2] (pp.35) it is assumed $f(K_{t+1}) = Z^t f(K_t) + (1-\delta)K_t$, where Z^t, $t = 0, 1, \ldots$ are random elements. We prefer a stochastic influence given by the assumptions i.8 or i.9.

Theorem 1. Let the assumption i.2, i.3, i.4, i.5 be fulfilled, then

$$P_{F^{\bar{\xi}^T}}\{\omega : \bar{\varphi}^R_{\mathcal{F}_\alpha}(K_0, \bar{F}^T, U) \leq \bar{\varphi}^R_{\mathcal{F}}(K_0, \bar{F}^T, U) \leq \bar{\varphi}^R_{\mathcal{F}^\alpha}(K_0, \bar{F}^T, U)\} \geq (1-\alpha)^T.$$

If moreover i.6, i.7, i.8 are fulfilled, then

$$P_{F^{\bar{\xi}^T}}\{\omega : \bar{\varphi}^R_{\mathcal{F}_\alpha(\varepsilon)}(K_0, \bar{F}^T, U) \leq \bar{\varphi}^R_{\mathcal{F}(\varepsilon)}(K_0, \bar{F}^T, U) \leq \bar{\varphi}^R_{\mathcal{F}^\alpha(\varepsilon)}(K_0, \bar{F}^T, U)\} \geq (1-\alpha)^T.$$

Remark.

1. Evidently if $\alpha = 0$, then the introduced bounds can be obtained by deterministic problems.
2. Replacing i.8 by i.9 we can obtain a similar result to the second assertion of Theorem 1.

3 A Note to Stability and Empirical Estimates

Employing multistage stochastic programming theory we can see that the problem (13) is a system of parametric one-stage stochastic problems in a relatively simple form. We utilize this fact to investigate the model. To this end, first, we recall some stability results for one–stage stochastic optimization.

3.1 One–Stage Case

Let $X \subset R^n$ be a nonempty compact set, $g_0(x, \zeta)$ be a function defined on $R^n \times R^1$, $\zeta := \zeta(\omega)$ be a random value defined on $(\Omega, \mathcal{S}, P)$. If we denote by F^ζ, P_{F^ζ}, Z_{F^ζ} the distribution function, the probability measure and the support corresponding to the random value ζ, then we can introduce simple one-stage stochastic programming problem in the following form:

Find

$$\varphi_0(F^\zeta, X) = \inf\{\mathsf{E}_{F^\zeta} g_0(x, \zeta) | x \in X\}.$$

Let $\mathcal{P}(R^1)$ denote the set of Borel probability measures on R^1 and let the system $\mathcal{M}_1(R^1)$ be defined by $\mathcal{M}_1(R^1) = \{P \in \mathcal{P}(R^1) : \int_{R^1} |z| P(dz) < \infty\}$. We introduce the following system of assumptions:

- A.1 $g_0(x, \zeta)$ is a uniformly continuous function on $X \times R^s$. Moreover, $g_0(x, \zeta)$ is for every $x \in X$ a Lipschitz function of ζ with the Lipschitz constant L not depending on x,
- A.2 $g_0(x, \zeta)$ is a uniformly continuous, bounded function on $X \times R^s$. Moreover $g_0(x, \zeta)$ is for every $\zeta \in Z_{F^\zeta}$ a Lipschitz function on X with the Lipschitz constant not depending on $\zeta \in Z_{F^\zeta}$,
- A.3 • $\{\zeta^i\}_{i=-\infty}^{\infty}$ is a sequence of independent random values corresponding to F^ζ,
  - F^ζ_N is an empirical distribution function determined by $\{\zeta^i\}_{i=1}^N$, $N = 1, 2, \ldots$,
- A.4 P_{F^ζ} is absolutely continuous w.r.t. the Lebesgue measure in R^1.

Proposition 2. [5] Let $P_{F^\zeta}, P_{G^\zeta} \in \mathcal{M}_1(R^1)$, X be a compact set. If A.1 is fulfilled, then

$$|\varphi_0(F^\zeta, X) - \varphi_0(G^\zeta, X)| \leq L \int_{R^1} |F^\zeta(z) - G^\zeta(z)| dz.$$

Proposition 3. [9] Let $P_{F^\zeta} \in \mathcal{M}_1(R^1)$, the assumptions A.1, A.3 be fulfilled, then

$$P\{\omega : \int_{R^1} |F^\zeta(z) - F^\zeta_N(z)| dz \longrightarrow_{N \longrightarrow \infty} 0\} = 1.$$

Let $t > 0$ and A.1 or A.2 and A.3, A.4 be fulfilled, then it is suitable to introduce $\gamma \in (0, 1/2)$ such that

$$P\{\omega : N^\gamma |\varphi_0(F^\zeta, X) - \varphi_0(F^\zeta_N, X)| > t\} \longrightarrow_{N \longrightarrow \infty} 0. \tag{20}$$

It follows from [7] that γ can depend on the distribution tails. If e.g. the tails are exponential, then the relation (20) is valid for $\gamma \in (0, 1/2)$. If we can assume only the Pareto tails, then some (weaker) results can also be proven.

3.2 Multistage Case

First, it follows from i.6 with either $\xi^t = \varepsilon^t H(\xi^{t-1})$, $t = \ldots, -1, 0, 1, \ldots$ or with $\xi^t = \varepsilon^t + H(\xi^{t-1})$, $t = \ldots, -1, 0, 1, \ldots$ and the relation (12) that (under rather general assumptions) the set $\bar{\mathcal{K}}_t((K_t\ \xi^t),\ t = 0, \ldots,$ are nonempty. Let us, furthermore, to assume:

A.4 $\bar{F}_t(N, K_t, \xi^t)$, $t = 0, \ldots, T$ are (for every N) Lipschitz functions of (K_t, ξ^t),

A.5 $u_t(C_t)$, $t = 0, \ldots, T$ are Lipschitz functions of C_t,

then for $F^\varepsilon, G^\varepsilon \in \mathcal{M}_1(R^1)$, employing the approach of [3], [4], [6] and rather general assumptions, we obtain two systems $\mathcal{F}(\varepsilon)$, $\mathcal{G}(\varepsilon)$ and $\bar{L} > 0$ such that

$$|\bar{\varphi}_{\mathcal{F}(\varepsilon)}(K_0, \bar{F}^T, U) - \bar{\varphi}_{\mathcal{G}(\varepsilon)}(K_0, \bar{F}^T, U)| \leq \bar{L} \int\limits_{R^1} |F^\varepsilon(z) - G^\varepsilon(z)| dz.$$

Furthermore, replacing G^ε by an empirical distribution functions F^ε_N determined by $\{\varepsilon^t\}_{t=-N}^{0}$ we obtain a system $\mathcal{F}_N(\varepsilon)$ and consequently also an empirical estimate $\bar{\varphi}_{\mathcal{F}_N(\varepsilon)}(K_0, \bar{F}^T, U)$ of $\bar{\varphi}_{\mathcal{F}(\varepsilon)}(K_0, \bar{F}^T, U)$. It follows from Proposition 3 and and the relation (20) that (under very "suitable" conditions) the properties (convergence rate) of the empirical estimates of $\bar{\varphi}_{\mathcal{F}}(K_0, F^T, U)$ are very similar to the properties of empirical estimates in one stage case.

4 Conclusion

In the contribution we have tried to give an analysis of the classical stochastic Ramsey model (with final horizon) employing stochastic programming approach. A relationship between dynamic programming and multistage problems has been already investigated (see e.g. [1]). It is very easy to see that the multistage stochastic programming problem corresponding to the Ramsey model is very suitable for this investigation. Especially, in the case of Markov dependence or "underlying" autoregressive sequences it gives possibility to construct confidence intervals for the optimal value (autoregressive case has been investigated for more general problem already e.g. in [6] and [8]). Evidently, the presented results can be generalized to more general stochastic programming problem, however this investigation is over possibilities of this contribution.

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